l = length of beam column
q = lateral load
w = lateral deflection of restrained beam = $a_1 f(x)$
x = axial distance along beam column
$f(x)$ = first fundamental lateral deflection mode
C = effective flexibility in axial direction = $(1/Kl) + (1/AE)$
EI, EA = bending and axial stiffness of cross section
F = restraining load
F_E = Euler load = $k\pi^2 EI/l^2$ [$k = 1$ (simple); $k = 4$(clamped)]
K = stiffness of axial restraint
M_0 = moment in beam unrestrained in axial direction
$\bar{\alpha T}$ = average axial strain of beam due to thermal expansion $= \left(\frac{1}{l}\right) \int_l^0 \alpha T dx$

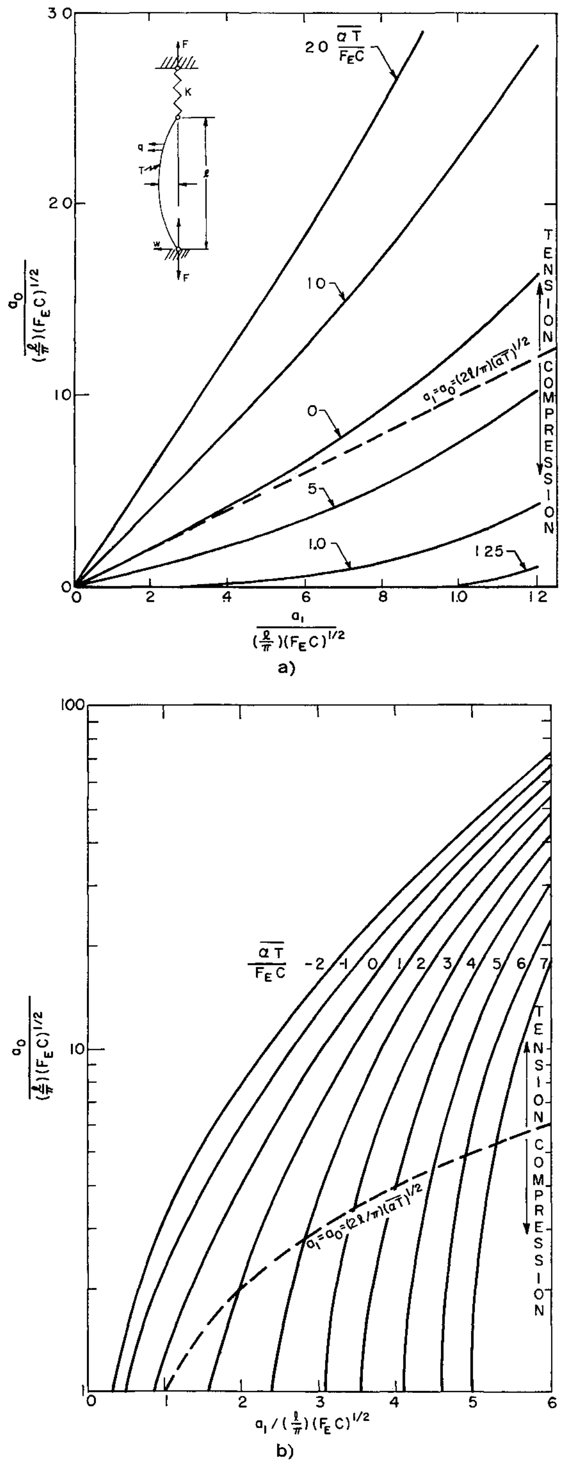

Fig 1 Unrestrained vs restrained lateral deflections of a beam column

A N approximate solution [Eq (1)] is presented in Figs 1a and 1b for the deflection of a restrained beam column of arbitrary cross section and boundary conditions subjected to lateral loads and temperature The figures can also be employed to obtain the restraining axial load and the stresses The solution is based upon the assumption that the beam is of constant cross-sectional stiffnesses and that the lateral deflection can be approximated by the first fundamental mode as a column

The physical solutions of the nondimensional equation

$$a_1[1 - (\bar{\alpha T}/F_E C)]/(l/\pi)(F_E C)^{1/2} + (\tfrac{1}{4})[a_1/(l/\pi)(F_E C)^{1/2}]^3 = a_0/(l/\pi)(F_E C)^{1/2} \quad (1)$$

exist in first quadrant Figure 1a presents small values of the final deflection parameter $[a_1/(l/\pi)(F_E C)^{1/2}]$ which differs only slightly from the unrestrained deflection parameter divided by $1 - (\bar{\alpha T}/F_E C)$ This solution is similar to the "linear" classical beam-column formula which ignores membrane action and where the lateral deflection grows as the reciprocal of the difference of the critical to applied strain (or load) Figure 1b presents larger "nonlinear" solutions $(a_1 \to [4a_0 F_E C(l/\pi)^2]^{1/3})$ where the membrane action becomes significant

Lateral loads and negative values of the axial expansion parameter $(\bar{\alpha T}/F_E C)$ cause the restraining load F to be positive, resulting in final lateral deflections which are smaller than the unrestrained deflections $(a_0/a_1 > 1)$ As the axial expansion parameter becomes increasingly positive, the restraining load will decrease and become negative (compression) The compressive loads will augment the unrestrained lateral deflections $(a_0/a_1 < 1)$ As the expansion parameter is further increased, the lateral deflections and compressive load will continue to increase—the deflections at an increasing rate, but the compressive load at a decreasing rate The increasing compressive load will approach, but never attain, the column buckling load $(-F_E)$

The axial load and moment is obtained from the following equations:

$$F = [(a_0/a_1) - 1]F_E > -F_E \quad (2)$$

$$M = M_0 - Fw = M_0 - Fa_1 f(x) \quad (3)$$

Linearized Analysis of the Pressure Waves in a Tank Undergoing an Acceleration

F Edward Ehlers*
Boeing Scientific Research Laboratories, Seattle, Wash

Introduction

TO obtain insight into the effects of acceleration on fluid flows, the linearized equations for the one-dimensional flow in an accelerated closed tank of compressible fluid are solved for the acceleration prescribed as a known function of time The wave pattern is described in detail for the flow induced by an instantaneous constant acceleration beginning at time $t = 0$

Equations of Motion for a Gas under Acceleration of Its Container

In a fixed inertial system the equations for the one-dimensional isentropic flow of an ideal gas take the form

Received September 16, 1963
* Staff Member Member AIAA

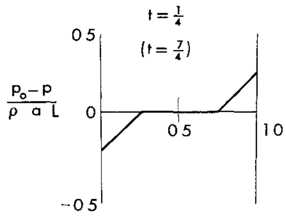

Fig 1

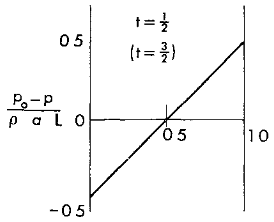

Fig 2

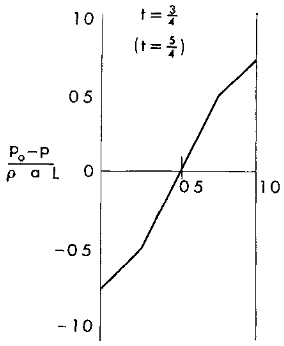

Fig 3

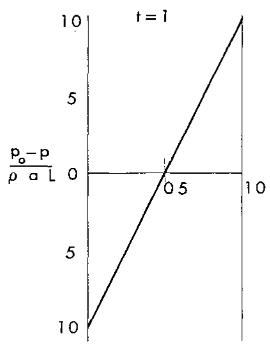

Fig 4

Figs 1–4 Pressure coefficient distribution in the tank for several values of time t

$$(\rho u')_{x}' = -\rho_t \tag{1}$$

$$u_t' + u'u'_x = -c^2\rho_x'/\rho \tag{2}$$

where ρ is the density, $c^2(\rho)$ the velocity of sound, u' the velocity, x' the position coordinate, and t the time

Consider a tank of length L moving with a velocity $u_0(t) = dx_0/dt$ and rewrite the equations for the velocity and position referred to an origin fixed on the tank Then, with

$$x = x' - x_0(t) \qquad u = u' - dx_0/dt \tag{3}$$

and

$$\rho_t = \rho_t - \rho_x dx_0/dt \qquad u_t' = u_t + d^2x_0/dt^2 - u_x dx_0/dt \tag{4}$$

Eqs (1) and (2) become

$$(\rho u)_x = -\rho_t \tag{5}$$

$$a_0(t) + u_t + uu_x = -c^2\rho_x/\rho \tag{6}$$

where $a_0(t) = d^2x_0/dt^2$ Linearization of Eqs (5) and (6) by considering only small perturbations from the initial undisturbed state of the air in the tank yields, finally, for the equations of motion,

$$\rho_t + u_x = 0 \tag{7}$$

and

$$a(t) + u_t + \rho_x = 0 \tag{8}$$

where u, ρ, x, and t are made dimensionless to the undisturbed state quantities c_0, ρ_0, L, and c_0/L, respectively, and $a = a_0(t)L/c_0^2$ Equation (7) is satisfied identically by the introduction of a stream function ψ defined by

$$\rho = -\psi_x \qquad u = \psi_t \tag{9}$$

Then Eq (8) reduces to the wave equation

$$\psi_{xx} - \psi_{tt} = a(t) \tag{10}$$

General Solution for the Stream Function

Equation (10) is solved conveniently by means of the Laplace transform For an initially undisturbed tank Eq (10) becomes

$$\bar{\psi}'' - p^2\bar{\psi} = \bar{a} \tag{11}$$

where the primes denote differentiation with respect to x The solution of this equation which satisfies the boundary conditions $\bar{\psi} = 0$ for vanishing velocity at the end walls, $x = 0$ and 1, is

$$\bar{\psi} = \bar{a}[(1 - \cosh p)\sinh px/\sinh p - (1 - \cosh px)]/p^2 \tag{12}$$

With the stream function for the unit step function acceleration denoted by $\psi_0(x,t)$, the inverse transform of Eq (12) takes the form

$$\psi(x,t) = \int_0^t a'(\tau)\psi_0(x,t-\tau)d\tau + a(0)\psi_0(x,t) \tag{13}$$

where $a'(t)$ is the dimensionless rate of acceleration of the tank

Stream Function ψ_0 for a Unit Constant Step in the Acceleration

Replacement of $\bar{a}$ by $1/p$ in Eq (12) and taking the inverse Laplace transform yields

$$\psi_0 = -t^2H(t)/2 + (t + x)^2H(t + x)/4 + (t - x)^2H(t - x)/4 + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt}(1 - \cosh p)\sinh px}{p^3 \sinh p}dp \tag{14}$$

where

$$H(z) = \begin{cases} 0 \text{ if } z < 0 \\ 1 \text{ if } z > 0 \end{cases}$$

When $t > x + 1$, the integration may be performed by the method of residues, and we obtain

$$\psi_0 = x(x-1)/2 + 2\sum_{n=0}^{\infty}\sin(2n+1)\pi(t+x)/(2n+1)^3\pi^3 - 2\sum_{n=0}^{\infty}\sin(2n+1)\pi(t-x)/(2n+1)^3\pi^3 \tag{15}$$

The pressure coefficient is found by differentiating Eq (15) with respect to x Summation of the resulting series[1] leads to

$$(p_0 - p)/\rho_0 a_0 L = -\rho = x - g(x + t) - g(x - t) \tag{16}$$

where

$$g(z) = \begin{cases} z/2 & 0 \leq z \leq 1 \\ 1 - z/2 & 1 \leq z \leq 2 \end{cases}$$

and $g(z)$ is periodic with period $\Delta z = 2$

A solution in simple waves is found by expanding the integrand of Eq (14) in powers of e^{-p} Thus the stream function becomes

$$\psi_0 = -t^2H(t)/2 + (t - x)^2H(t - x)/2 - \sum_{n=1}^{\infty}(t + x - n)^2H(t + x - n)(-1)^n/2 + \sum_{n=1}^{\infty}(t - x - n)^2H(t - x - n)(-1)^n/2 \tag{17}$$

For any given x, $0 < x < 1$, and time t, only a finite number of terms are nonzero, and so convergence of the expansion in e^{-p} is not required

Discussion of Results

The distribution in the tank of the pressure coefficient, $(p_0 - p)/\rho_0 a_0 L$, as computed from the simple wave solution, is shown in Figs 1–4 The unit for t is the time for an acoustic wave to travel the length of the tank We see that beginning at $t = 0$ a compression wave starts at $x = 0$ and propagates to the right, whereas an expansion wave starts at $x = 1$ and propagates to the left The pattern has a period $\Delta t = 2$, or the time required for an acoustic wave to travel to the opposite end of the tank and return

Computation for $t = 2$ to 4 of Eq (16) shows that it is valid for all $t > 0$, and the restriction $t > x + 1$ may be removed if $g(z)$ is extended to negative z by its periodicity Note that the pressure in the center of the tank is not influenced by the acceleration

For the incompressible fluid, we see from Eq (7) and the boundary conditions that $u \equiv 0$, and the linearized momentum equation after integration becomes $(p - p_0) = -\rho_0 a_0 x$ Thus the acceleration acts upon the fluid as a time varying gravitational field We easily recognize that in Eq (16) the solution is separated into 3 parts: the first represents the hydrostatic pressure term, the other two terms represent right and left propagating waves

Reference

[1] Hodgman, C D, *C R C Standard Mathematical Tables* (Chemical Rubber Publishing Company, Cleveland, Ohio 1959), 12th ed, p 378

Approximate Solution to Flux Concentration by Hydromagnetic Flow

Jerome M Baker*
Marquardt Corporation, Van Nuys, Calif

Nomenclature

B = magnetic flux density
E = electric field intensity
Ei = exponential integral $\left(\int_{-\infty}^{-x} t^{-1}e^{t}dt\right)$ tabulated in Ref 6
f = function defined by Eq (10)
J = current density
L = characteristic length equal to outer radius of annulus
p = pressure
R_m = magnetic Reynolds number defined by Eq (15)
r = radius
u = velocity component in radial direction
V = velocity vector
z = axial direction
μ_0 = magnetic permeability ($4\pi \times 10^{-7}$ henry/m)
ν = kinematic viscosity
ξ = dimensionless radius
ρ = fluid density
σ = fluid electrical conductivity
φ = azimuthal direction

Introduction

FLUX concentration by the interaction of a flowing conductor with an applied magnetic field has been investigated previously [1-3] Results were obtained either by numerical methods or by rather complicated mathematical expressions which were limited to low magnetic Reynolds numbers This note presents a simple approximate solution which is valid for all magnetic Reynolds numbers and is in good agreement with the previously presented numerical results

Received September 16, 1963

* Senior Research Engineer, ASTRO; presently Member of the Technical Staff, Aerospace Corporation, Los Angeles, Calif Member AIAA

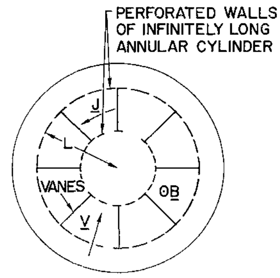

Fig 1 Hydromagnet geometry

The assumed geometry of the hydromagnet is shown in Fig 1 A conducting fluid flows radially inward between vanes in an infinitely long annular cylinder An axially applied magnetic field induces an azimuthal electric current, giving rise to an induced magnetic field in the same direction as the applied field The ultimate operating characteristics of the device depend on the properties of the conducting fluid, the magnetic Reynolds number, and the ratio of the internal and external radii of the annulus

Analysis

To analyze the flow, a cylindrical coordinate system with the z axis coincident with the hydromagnet is assumed The conducting fluid is assumed to be incompressible, and the fluid flow between the vanes is purely radial It also is assumed that the magnetic field has a z component only

Under these restrictions, the momentum equations in the r and φ directions are, respectively,[3]

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B})_r + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \tag{1}$$

$$0 = -\frac{1}{r\rho}\frac{\partial p}{\partial \varphi} + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B})_\varphi + 2\frac{\nu}{r^2}\frac{\partial u}{\partial \varphi} \tag{2}$$

For steady-state operation, Maxwell's equations in rationalized mks units are

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{3}$$

$$\nabla\ \mathbf{B} = 0 \tag{4}$$

$$\nabla \times \mathbf{E} = 0 \tag{5}$$

Finally, Ohm's law is given by

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \tag{6}$$

The electromagnetic body force per unit fluid volume can be written, after some vector manipulation, as

$$\mathbf{J}\times\mathbf{B} = -\frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(\mathbf{B}\ \nabla)\mathbf{B} \tag{7}$$

In view of the assumption that the magnetic field has a z component only, the last term in Eq (7) can be shown to vanish Then, use of the resulting expression in Eqs (1) and (2) yields

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \tag{8}$$

$$0 = -\frac{1}{\rho r}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \frac{2\nu}{r^2}\frac{\partial u}{\partial \varphi} \tag{9}$$

The equation of continuity for purely radial, incompressible flow reduces to

$$u = f(\varphi)/r \tag{10}$$

Substitution of Eq (10) into (8) and (9) gives

$$-\frac{f^2}{r^3} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \frac{\nu}{r^3}\frac{\partial^2 f}{\partial \varphi^2} \tag{11}$$